# SNAP LECTURE 3: QUANTUM ERGODICITY

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In this Lecture, we assume that the geodesic flow of $(M, g)$ is ergodic. Ergodicity of $G^t$ means that Liouville measure $d\mu_L$ is an ergodic measure for $G^t$ on $S^*M$, i.e. an extreme point of $\mathcal{M}_I$. Equivalently, any $G^t$-invariant set has Liouville measure zero or one. Ergodicity is a spectral property of the operator $V^t f(\zeta) = f(G^t(\zeta))$ on $L^2(S^*M, d\mu_L)$, namely that $V^t$ has 1 as an eigenvalue of multiplicity one, i.e. only invariant $L^2$ functions (with respect to Liouville measure) are the constant functions.

In this case, there is a general result which originated in the work of A. I. Schnirelman and was developed into the following theorem by S. Zelditch, Y. Colin de Verdière on manifolds without boundary and by P. Gérard-E. Leichtnam and S. Zelditch-M. Zworski on manifolds with boundary. The following variance theorem is a combination of [Ze90, Su].

Let $\omega$ denote the Liouville measure on $S^*M$, i.e. the Leray measure $\frac{(dx\wedge d\xi)^n}{dH}$ where $(dx \wedge d\xi)^n$ is the symplectic volume measure and where $H(x, \xi) = |\xi|_g$. We view $\omega$ as a linear functional on $C(S^*M)$. Also, let $N(\lambda) = \#\{j : \lambda_j \leq \lambda\}$.

VAR **Theorem 0.1.** *Let $(M, g)$ be a compact Riemannian manifold (possibly with boundary), and let $\{\lambda_j, \varphi_j\}$ be the spectral data of its Laplacian $-\Delta$. Then the geodesic flow $G^t$ is ergodic on $(S^*M, d\mu_L)$ if and only if, for every $A \in \Psi^0(M)$, we have:*

(i) $\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |(A\varphi_j, \varphi_j) - \omega(A)|^2 = 0.$

(ii) *For all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\limsup_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\substack{j\neq k:\ \lambda_j,\lambda_k\leq\lambda \\ |\lambda_j-\lambda_k|<\delta}} |(A\varphi_j, \varphi_k)|^2 < \varepsilon.$$

The 'if' direction was proved in [Ze90]. The fact that the proof can be reversed to prove the converse direction was observed by Sunada in [Su].

The diagonal result may be interpreted as a variance result for the local Weyl law. Since all the terms are positive, the asymptotic is equivalent to the existence of a s subsequence $\{\varphi_{j_k}\}$ of eigenfunctions whose indices $j_k$ have counting density one for which $\langle A\varphi_{j_k}, \varphi_{j_k}\rangle \to \omega(A)$ for any $A \in \Psi^0(M)$. As above, such a sequence of eigenfunctions is called ergodic. One can sharpen the results by averaging over eigenvalues in the shorter interval $[\lambda, \lambda+1]$ rather than in $[0, \lambda]$.

The first statement (i) is essentially a convexity result. It remains true if one replaces the square by any convex function $F$ on the spectrum of $A$,

CONVEX (1)
$$\frac{1}{N(E)} \sum_{\lambda_j\leq E} F(\langle A\varphi_k, \varphi_k\rangle - \omega(A)) \to 0.$$

The basic QE (quantum ergodicity) theorem is the following:

QE THEOREM **0.2.** *Let $(M, g)$ be a compact Riemannian manifold (possibly with boundary), and let $\{\lambda_j, \varphi_j\}$ be the spectral data of its Laplacian $-\Delta$. Then, if the geodesic flow $G^t$ is ergodic on $(S^*M, d\mu_L)$, there exists a subsequence $\mathcal{S}$ of density one, $D^*(\mathcal{S}) = 1$ such that*

$$\lim_{j\to\infty, j\in\mathcal{S}} \langle A\varphi_j, \varphi_j\rangle \to \omega(A)$$

*for all $A \in \Psi^0(M)$.*

Density one means that $\frac{1}{N(\lambda)}\#\{j : \lambda_j \leq \lambda, j \in \mathcal{S}\} \to 1$ as $\lambda \to \infty$.

0.1. **Quantum ergodicity in terms of operator time and space averages.** The diagonal variance asymptotics may be interpreted as a relation between operator time and space averages.

TA **Definition** *Let $A \in \Psi^0$ be an observable and define its time average to be:*

$$\langle A\rangle := \lim_{T\to\infty} \langle A\rangle_T,$$

*where*

$$\langle A\rangle_T := \frac{1}{2T}\int_{-T}^{T} U^t A U^{-t} dt$$

*and its space average to be scalar operator*

$$\omega(A) \cdot I$$

Then Theorem 0.2 (1) is (almost) equivalent to,

$$\langle A\rangle = \omega(A)I + K, \qquad \text{where} \qquad \lim_{\lambda\to\infty} \omega_\lambda(K^*K) \to 0, \tag{2}$$

where $\omega_\lambda(A) = \frac{1}{N(\lambda)} Tr E(\lambda) A$. Thus, the time average equals the space average plus a term $K$ which is semi-classically small in the sense that its Hilbert-Schmidt norm square $||E_\lambda K||^2_{HS}$ in the span of the eigenfunctions of eigenvalue $\leq \lambda$ is $o(N(\lambda))$.

This is not exactly equivalent to Theorem 0.2 (1) since it is independent of the choice of orthonormal basis, while the previous result depends on the choice of basis. However, when all eigenvalues have multiplicity one, then the two are equivalent. To see the equivalence, note that $\langle A\rangle$ commutes with $\sqrt{-\Delta}$ and hence is diagonal in the basis $\{\varphi_j\}$ of joint eigenfunctions of $\langle A\rangle$ and of $U_t$. Hence $K$ is the diagonal matrix with entries $\langle A\varphi_k, \varphi_k\rangle - \omega(A)$. The condition is therefore equivalent to

$$\lim_{E\to\infty} \frac{1}{N(E)} \sum_{\lambda_j \leq E} |\langle A\varphi_k, \varphi_k\rangle - \omega(A)|^2 = 0.$$

0.2. **Heuristic proof of Theorem 0.2 (i).** There is a simple picture of eigenfunction states which makes Theorem 0.2 seem obvious. Justifying the picture is more difficult than the formal proof below but the reader may find it illuminating and convincing.

First, one should re-formulate the ergodicity of $G^t$ as a property of the Liouville measure $d\mu_L$: ergodicity is equivalent to the statement $d\mu_L$ is an extreme point of the compact convex set $\mathcal{M}_I$. Moreover, it implies that the Liouville state $\omega$ on $\Psi^0(M)$ is an extreme point of the compact convex set $\mathcal{E}_{\mathbb{R}}$ of invariant states for $\alpha_t(A) = U^{-t}AU^t$; see [Ru] for background. But the local Weyl law says that $\omega$ is also the limit of the convex combination $\frac{1}{N(E)}\sum_{\lambda_j \leq E} \rho_j$. An extreme point cannot be written as a convex combination of other states unless all the states in the combination are equal to it. In our case, $\omega$ is only a limit of convex combinations so it need not (and does not) equal each term. However, almost all terms in the sequence must tend to $\omega$, and that is equivalent to (1).

One could make this argument rigorous by considering whether Liouville measure is an *exposed point* of $\mathcal{E}_I$ and $\mathcal{M}_I$. Namely, is there a linear functional $\Lambda$ which is equal to zero at $\omega$ and is $< 0$ everywhere else on $\mathcal{E}_I$? If so, the fact that $\frac{1}{N(E)}\sum_{\lambda_j \leq E} \Lambda(\rho_j) \to 0$ implies that $\Lambda(\rho_j) \to 0$ for a subsequence of density one. For one gets an obvious contradiction if $\Lambda(\rho_{j_k}) \leq -\varepsilon < 0$ for some $\varepsilon > 0$ and a subsequence of positive density. But then $\rho_{j_k} \to \omega$ since $\omega$ is the unique state with $\Lambda(\rho) = 0$.

In [?] it is proved that Liouville measure (or any ergodic measure) is exposed in $\mathcal{M}_I$. It is stated in the following form: For any ergodic invariant probability measure $\mu$, there exists a continuous function $f$ on $S^*M$ so that $\mu$ is the unique $f$-maximizing measure in the sense that

$$\int f d\mu = \sup \left\{ \int f dm : m \in \mathcal{M}_I \right\}.$$

To complete the proof, one would need to show that the extreme point $\omega$ is exposed in $\mathcal{E}_I$ for the C* algebra defined by the norm-closure of $\Psi^0(M)$.

## 1. Weyl law and local Weyl law

A fundamental and classical result in spectral asymptotics is Weyl's law on counting eigenvalues:

WL

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\} = \frac{|B_n|}{(2\pi)^n} Vol(M,g)\lambda^n + O(\lambda^{n-1}). \tag{3}$$

Here, $|B_n|$ is the Euclidean volume of the unit ball and $Vol(M,g)$ is the volume of $M$ with respect to the metric $g$. An equivalent formula which emphasizes the correspondence between classical and quantum mechanics is:

$$Tr E_\lambda = \frac{Vol(|\xi|_g \leq \lambda)}{(2\pi)^n}, \tag{4}$$

where $Vol$ is the symplectic volume measure relative to the natural symplectic form $\sum_{j=1}^n dx_j \wedge d\xi_j$ on $T^*M$. Thus, the dimension of the space where $H = \sqrt{\Delta}$ is $\leq \lambda$ is asymptotically the volume where its symbol $|\xi|_g \leq \lambda$.

The remainder term in Weyl's law is sharp on the standard sphere, where all geodesics are periodic, but is not sharp on $(M,g)$ for which the set of periodic geodesics has measure zero (Duistermaat-Guillemin, Ivrii). When the set of periodic geodesics, has measure zero (as is the case for ergodic systems), one has

DGI

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\} = \frac{|B_n|}{(2\pi)^n} Vol(M,g)\lambda^n + o(\lambda^{n-1}). \tag{5}$$

The remainder is then of small order than the derivative of the principal term, and one then has asymptotics in shorter intervals:

DGIshort

$$N([\lambda, \lambda+1]) = \#\{j : \lambda_j \in [\lambda, \lambda+1]\} = n\frac{|B_n|}{(2\pi)^n} Vol(M,g)\lambda^{n-1} + o(\lambda^{n-1}). \tag{6}$$

Physicists tend to write $\lambda \sim h^{-1}$ and to average over intervals of this width. Then mean spacing between the eigenvalues in this interval is $\sim C_n Vol(M,g)^{-1} \lambda^{-(n-1)}$, where $C_n$ is a constant depending on the dimension.

An important generalization is the *local Weyl law* concerning the traces $TrAE(\lambda)$ where $A \in \Psi^m(M)$. It asserts that

LWL (7)
$$\sum_{\lambda_j \leq \lambda} \langle A\varphi_j, \varphi_j\rangle = \frac{1}{(2\pi)^n} \int_{B^*M} \sigma_A dx d\xi) \lambda^n + O(\lambda^{n-1}).$$

There is also a pointwise local Weyl law:

PLWL (8)
$$\sum_{\lambda_j \leq \lambda} |\varphi_j(x)|^2 = \frac{1}{(2\pi)^n} |B^n| \lambda^n + R(\lambda, x),$$

where $R(\lambda, x) = O(\lambda^{n-1})$ uniformly in $x$. Again, when the periodic geodesics form a set of measure zero in $S^*M$, one could average over the shorter interval $[\lambda, \lambda+1]$. Combining the Weyl and local Weyl law, we find the surface average of $\sigma_A$ is a limit of traces:

OMEGA (9)
$$\begin{aligned} \omega(A) &:= \frac{1}{\mu(S^*M)} \int_{S^*M} \sigma_A d\mu \\ &= \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle A\varphi_j, \varphi_j\rangle \end{aligned}$$

Here, $\mu$ is the *Liouville measure* on $S^*M$, i.e. the surface measure $d\mu = \frac{dxd\xi}{dH}$ induced by the Hamiltonian $H = |\xi|_g$ and by the symplectic volume measure $dxd\xi$ on $T^*M$.

## 2. Sketch of Proof of Theorems 0.1 and 0.2 (i)

**2.1. Proof of Theorem 0.1.** We now sketch the proof of Theorem 0.1. By time averaging (Definition 0.1), we have

(10)
$$\sum_{\lambda_j \leq E} |\langle A\varphi_k, \varphi_k\rangle - \omega(A)|^2 = \sum_{\lambda_j \leq E} |\langle\langle A\rangle_T - \omega(A)\varphi_k, \varphi_k\rangle|^2.$$

We then apply the Schwartz inequality to get:

(11)
$$\begin{aligned} \textstyle\sum_{\lambda_j \leq E} |\langle(\langle A\rangle_T - \omega(A))\, \varphi_k, \varphi_k\rangle|^2 &\leq \textstyle\sum_{\lambda_j \leq E} \langle(\langle A\rangle_T - \omega(A))^2\, \varphi_k, \varphi_k\rangle \\ &= \mathrm{Tr}\, (\Pi_E [\langle A\rangle_T - \omega(A)]^2 \Pi_E) \\ &=: \omega_E([\langle A\rangle_T - \omega(A)]^2). \end{aligned}$$

Above, $\Pi_E$ is the spectral projection for $\hat{H}$ corresponding to the interval $[0, E]$. By the local Weyl law, $\omega_E \to \omega$. Hence,

$$\lim_{E \to \infty} \frac{1}{N(E)} \sum_{\lambda_j \leq E} |\langle A\varphi_k, \varphi_k\rangle - \omega(A)|^2 \leq \int_{\{H=1\}} |\langle\sigma_A\rangle_T - \omega(A)|^2 d\mu_L.$$

As $T \to \infty$ the right side approaches $\varphi(0)$ by the $L^2$ von Neumann mean ergodic theorem. Since the left hand side is independent of $T$, this implies that

$$\lim_{E\to\infty} \frac{1}{N(E)} \sum_{\lambda_j \leq E} |\langle A\varphi_k, \varphi_k\rangle - \omega(A)|^2 = 0.$$

□

It is useful to note that the same result is true if $F(x) = x^2$ is replaced by any convex function. The generalization to convex functions is useful in obtaining rates of quantum ergodicity [Ze96, Schu, Schu2, AR12]. The rates were used to improve various results on nodal sets and $L^p$ norms on balls of shrinking radius $r(\lambda) = \frac{1}{(\log\lambda)^\gamma}$ for certain $\gamma > 0$ in [Han15, HeR16].

By time averaging, we have

$$\sum_{\lambda_j \leq E} F(\langle A\varphi_k, \varphi_k\rangle - \omega(A)) = \sum_{\lambda_j \leq E} F(\langle\langle A\rangle_T - \omega(A)\varphi_k, \varphi_k\rangle). \tag{12}$$

We then apply the Peierls–Bogoliubov inequality

$$\sum_{j=1}^{n} F((B\varphi_j, \varphi_j)) \leq \operatorname{Tr} F(B)$$

with $B = \Pi_E[\langle A\rangle_T - \omega(A)]\Pi_E$ to get:

$$\sum_{\lambda_j \leq E} F(\langle\langle A\rangle_T - \omega(A)\varphi_k, \varphi_k\rangle) \leq \operatorname{Tr} F(\Pi_E[\langle A\rangle_T - \omega(A)]\Pi_E). \tag{13}$$

Above, $\Pi_E$ is the spectral projection for $\hat{H}$ corresponding to the interval $[0, E]$. By the Berezin inequality (if $F(0) = 0$):

$$\begin{aligned} \frac{1}{N(E)} \operatorname{Tr} F(\Pi_E[\langle A\rangle_T - \omega(A)]\Pi_E) &\leq \frac{1}{N(E)} \operatorname{Tr} \Pi_E F([\langle A\rangle_T - \omega(A)])\Pi_E \\ &= \omega_E(\varphi(\langle A\rangle_T - \omega(A))). \end{aligned}$$

As long as $F$ is smooth, $F(\langle A\rangle_T - \omega(A))$ is a pseudodifferential operator of order zero with principal symbol $F(\langle\sigma_A\rangle_T - \omega(A))$. By the assumption that $\omega_E \to \omega$ we get

$$\lim_{E\to\infty} \frac{1}{N(E)} \sum_{\lambda_j \leq E} F(\langle A\varphi_k, \varphi_k\rangle - \omega(A)) \leq \int_{\{H=1\}} F(\langle\sigma_A\rangle_T - \omega(A)) d\mu_L.$$

As $T \to \infty$ the right side approaches $\varphi(0)$ by the dominated convergence theorem and by Birkhoff's ergodic theorem. Since the left hand side is independent of $T$, this implies that

$$\lim_{E\to\infty} \frac{1}{N(E)} \sum_{\lambda_j \leq E} F(\langle A\varphi_k, \varphi_k\rangle - \omega(A)) = 0$$

for any smooth convex $F$ on $\operatorname{Spec}(A)$ with $F(0) = 0$. □

2.2. **Proof of Theorem 0.2.** The proof is an application of Chebychev's inequality and a diagonal argument to the variance result of Theorem 0.1.

First we show that for any $A \in \Psi^0(M)$ there is a density one subsequence $\mathcal{S}_A$ such that $\langle A\varphi_j, \varphi_j\rangle \to \omega(A)$. For fixed $\lambda$, we put a probability measure $\mathbf{P}_\lambda$ on $\{j : \lambda_j \leq \lambda\}$ by $\frac{1}{N(\lambda)}\sum_{j:\lambda_j\leq\lambda}\delta_{\lambda_j}$. Denote the expected value by $\mathbf{E}_\lambda$ and define the random variable $X_\lambda$ on $\{j : \lambda_j \leq \lambda\}$ by $X_\lambda(j) = |\langle A\varphi_j, \varphi_j\rangle - \omega(A)|^2$. Then,

$$\mathbf{E}_\lambda X_\lambda = \sum_{j:\lambda_j\leq\lambda} |\langle A\varphi_j, \varphi_j\rangle - \omega(A)|^2 =: \varepsilon^2(\lambda).$$

By Theorem 0.1, $\varepsilon(\lambda) \to 0$ as $\lambda \to \infty$. Let

$$\Gamma(\lambda) = \{j : \lambda_j \leq \lambda, |\langle A\varphi_j, \varphi_j\rangle - \omega(A)|^2 \geq \varepsilon(\lambda)\}, \;\; \Lambda(\lambda) := \{j : \lambda_j \leq \lambda\}\backslash\Gamma(\lambda).$$

By Chebyshev's inequality,

$$\mathbf{P}_\lambda(\Gamma_\lambda) \leq \frac{\varepsilon^2(\lambda)}{\varepsilon(\lambda)} = \varepsilon(\lambda).$$

Then,

$$\mathbf{P}_\lambda(\Lambda(\lambda)) = 1 - \frac{\#\Gamma(\lambda)}{N(\lambda)} \geq 1 - \varepsilon(\lambda).$$

We then dissect $\mathbb{R}_+$ into intervals $[N, N+1]$ and let $\Lambda_N(\lambda) = \Lambda(\lambda) \cap [N, N+1]$. We then define $\Lambda_\infty$ by $\Lambda_\infty \cap [N, N+1] = \Lambda_N \cap [N, N+1]$.

The sets $\Gamma(\lambda), \Lambda(\lambda)$ depend on $A$. We now use a diagonalization argument to obtain a subsequence of density one which works for all $A$.

Since $\Psi^0(M)$ is separable, there exists a countable dense subset $\{A_j\}$ of the unit ball of $\Psi^0(M)$. For each $j$, let $\mathcal{S}_j \subset \mathbb{N}$ be a density one subsequence $\mathcal{S}_A$ such that $\langle A\varphi_k, \varphi_k\rangle \to \omega(A)$ (with $k \in \mathcal{S}_A$) for $A_j$. We may assume $\mathcal{S}_j \subset \mathcal{S}_{j+1}$. Then choose $N_j$ so that

$$\frac{1}{N}\#\{k \in \mathcal{S}_j : k \leq N\} \geq 1 - 2^{-j} \text{ for } N \geq N_j \; .$$

Let $\mathcal{S}_\infty$ be the subsequence defined by

$$\mathcal{S}_\infty \cap [N_j, N_{j+1}] = \mathcal{S}_j \cap [N_j, N_{j+1}].$$

Then $\mathcal{S}_\infty$ is of density one and

$$\lim_{k\to\infty, k\in\mathcal{S}_\infty} \rho_k(A) = \omega(A)$$

for all $A \in \Psi^0(M)$, since it holds for the set $\{A_j\}$ and since $\{A_j\}$ is dense in the unit ball.

*Proof.* □

2.3. **Rate of quantum ergodicity and mixing.** A quantitative refinement of quantum ergodicity is to ask at what rate the sums in Theorem 0.2(1) tend to zero, i.e. to establish a rate of quantum ergodicity. In the off-diagonal case one may view $|\langle A\varphi_i, \varphi_j\rangle|^2$ as analogous to $|\langle A\varphi_j, \varphi_j) - \omega(A)|^2$. However, the sums in (15) are double sums while those of (??) are single. One may also average over the shorter intervals $[\lambda, \lambda + 1]$.

The only rigorous result valid on general Riemannian manifolds with hyperbolic geodesic flow is the logarithmic decay. In [Ze94] is proved:

Theorem **2.1.** *For any $(M, g)$ with hyperbolic geodesic flow,*

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |(A\varphi_j, \varphi_j) - \omega(A)|^{2p} = O\left(\frac{1}{(\log \lambda)^p}\right).$$

The proof uses the central limit theorem for geodesic flows of M. Ratner. The logarithm reflects the exponential blow up in time of remainder estimates for traces involving the wave group associated to hyperbolic flows. It would be surprising if the logarithmic decay is sharp for Laplacians. It was shown by R. Schubert [Schu] that the estimate is sharp in the case of two-dimensional hyperbolic quantum cat maps. Hence the estimate cannot be improved by semi-classical arguments that hold in both settings.

2.4. **QUE in terms of time and space averages.** The quantum unique ergodicity problem (the term is due to Rudnick-Sarnak [RS]) is the following:

Problem **2.2.** *Suppose the geodesic flow $G^t$ of $(M, g)$ is ergodic on $S^*M$. Is the operator $K$ in*

$$\langle A \rangle = \omega(A) + K$$

*a compact operator? Equivalently is $\mathcal{Q} = \{d\mu_L\}$? In this case, $\sqrt{-\Delta}$ is said to be QUE (quantum uniquely ergodic)*

Compactness of $K$ implies that $\langle K\varphi_k, \varphi_k \rangle \to 0$, hence $\langle A\varphi_k, \varphi_k \rangle \to \omega(A)$ along the entire sequence.

Rudnick-Sarnak conjectured that $\Delta$ of negatively curved manifolds are QUE, i.e. that for any orthonormal basis of eigenfunctions, the Liouville measure is the only quantum limit [RS].

2.5. **Converse QE.** So far we have not mentioned Theorem 0.2 (2). An interesting open problem is the extent to which (2) is actually necessary for the equivalence to classical ergodicity.

Problem **2.3.** *Suppose that $\sqrt{-\Delta}$ is quantum ergodic in the sense that (1) holds in Theorem 0.2. What are the properties of the geodesic flow $G^t$. Is it ergodic (in the generic case)?*

In the larger class of Schrödinger operators, there is a simple example of a Hamiltonian system which is quantum ergodic but not classically ergodic: namely, a Schrödinger operator with a symmetric double well potential $W$. That is, $W$ is a $W$ shaped potential with two wells and a $\mathbb{Z}_2$ symmetry exchanging the wells. The low energy levels consist of two connected components interchanged by the symmetry, and hence the classical Hamiltonian flow is not ergodic. However, all eigenfunctions of the Schrödinger operator $-\frac{d^2}{dx^2} + W$ are either even or odd and thus have the same mass in both wells. It is easy to see that the quantum Hamiltonian is quantum ergodic.

Recently, B. Gutkin [Gut] has given a two dimensional example of a domain with boundary which is quantum ergodic but not classically ergodic and which is a two dimensional analogue of a double well potential. The domain is a so-called hippodrome (race-track) stadium. Similarly to the double well potential, there are two invariant sets interchanged by a $\mathbb{Z}_2$ symmetry. They correspond to the two orientations with which the race could occur. Hence the classical billiard flow on the domain is not ergodic. After dividing by the $\mathbb{Z}^2$ symmetry the hippodrome has ergodic billiards, hence by Theorem 0.2, the quotient domain is quantum ergodic. But the The eigenfunctions are again either even or odd. Hence the hippodrome is quantum ergodic but not classically ergodic.

Little is known about converse quantum ergodicity in the abscence of symmetry. It is known that if there exists an open set in $S^*M$ filled by periodic orbits, then the Laplacian cannot be quantum ergodic (see [MOZ] for recent results and references). But it is not even known at this time whether KAM systems, which have Cantor-like invariant sets of positive measure, are not quantum ergodic. It is known that there exist a positive proportion of approximate eigenfunctions (quasi-modes) which localize on the invariant tori, but it has not been proved that a positive proportion of actual eigenfunctions have this localization property.

## QWMS 3. Quantum weak mixing

There are parallel results on quantizations of weak-mixing geodesic flows. We recall that the geodesic flow of $(M, g)$ is weak mixing if the operator $V^t$ has purely continuous spectrum on the orthogonal complement of the constant functions in $L^2(S^*M, d\mu_L)$. The following is proved in [?]:

QWM **Theorem 3.1.** *The geodesic flow $G^t$ of $(M, g)$ is weak mixing if and only if the conditions (1)-(2) of Theorem 0.2 hold and additionally, for any $A \in \Psi^o(M)$,*

$$(\forall \varepsilon)(\exists \delta) \limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\substack{j \neq k: \lambda_j, \lambda_k \leq \lambda \\ |\lambda_j - \lambda_k - \tau| < \delta}} |(A\varphi_j, \varphi_k)|^2 < \varepsilon \qquad (\forall \tau \in \mathbb{R}).$$

The restriction $j \neq k$ is of course redundant unless $\tau = 0$, in which case the statement coincides with quantum ergodicity. This result follows from the general asymptotic formula, valid for any compact Riemannian manifold $(M, g)$, that

QMF (14)

$$\frac{1}{N(\lambda)} \sum_{\substack{i \neq j \\ \lambda_i, \lambda_j \leq \lambda}} |\langle A\varphi_i, \varphi_j \rangle|^2 \left| \frac{\sin T(\lambda_i - \lambda_j - \tau)}{T(\lambda_i - \lambda_j - \tau)} \right|^2 \sim \left\| \frac{1}{2T} \int_{-T}^{T} e^{it\tau} V_t(\sigma_A) \right\|_2^2 - \left| \frac{\sin T\tau}{T\tau} \right|^2 \omega(A)^2.$$

In the case of weak-mixing geodesic flows, the right hand side $\to 0$ as $T \to \infty$. As with diagonal sums, the sharper result is true where one averages over the short intervals $[\lambda, \lambda + 1]$.

Theorem 3.1 is based on expressing the spectral measures of the geodesic flow in terms of matrix elements. The main limit formula is:

SPECMEAS (15)
$$\int_{\tau - \varepsilon}^{\tau + \varepsilon} d\mu_{\sigma_A} := \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{i,j:\ \lambda_j \leq \lambda,\ |\lambda_i - \lambda_j - \tau| < \varepsilon} |\langle A\varphi_i, \varphi_j \rangle|^2 \ ,$$

where $d\mu_{\sigma_A}$ is the spectral measure for the geodesic flow corresponding to the principal symbol of $A$, $\sigma_A \in C^\infty(S^*M, d\mu_L)$. Recall that the spectral measure of $V^t$ corresponding to $f \in L^2$ is the measure $d\mu_f$ defined by

$$\langle V^t f, f \rangle_{L^2(S^*M)} = \int_{\mathbb{R}} e^{it\tau} d\mu_f(\tau) \ .$$

3.1. **Open problems on quantum ergodicity.** As a test of how much is known about quantum ergodicity, we offer the following open problems. In each, the motivating case is where the geodesic flow is ergodic or that the eigenfunctions are ergodic.

(1) Suppose that $\varphi_{j_k}^2 \to 1$ in the weak* sense on $C(M)$. Prove that $\varphi_{j_k}$ does not tend to 1 in the strong sense. For instance show that $\liminf_{k\to\infty} ||\varphi_{j_k}^2 - 1||_{L^1} > 0$.

(2) Similarly show that $\liminf_{k\to\infty} ||\lambda_{j_k}^{-2}|\nabla\varphi_{j_k}|^2 - 1|_{L^1} > 0$.

(3) Are the measures $\varphi_j^2$ uniformly integrable? Show that $\|\varphi_{j_k}\|_{L^p} \to \infty$ at least along a density one subsequence for each $p > 2$.

(4) Does there exist $\delta > 0$ so that $\|\varphi_{j_k}\|_{L^1} \geq \delta$? This is false for Gaussian beams but might occur in the ergodic case. Find a good lower bound.

(5) Is there a subsequence of density one so that $\int_E \varphi_{j_k}^2 dV \to \frac{|E|}{|M|}$ for every Borel set? Here $|E|$ is its Liouville measure. This is true by the Portmanteau theorem if $\partial E$ has Liouville measure zero. For each Borel set there does exist a subsequence of density one with this property, but $L^\infty$ is not separable and one cannot use a diagonalization argument to find a density one subsequence.

(6) In weak* limit formulae one holds the set fixed. But for quantum ergodic eigenfunctions one might expect the 'mean values' to dominante. Does exist a constant $M$ so that $\int_{|\varphi_{j_k}|\leq M} V\varphi_{j_k}^2 dV_g \to \int_M V$? Note that if $p(\lambda) \to \infty$ then $\int_{|\varphi_{j_k}|\geq p(\lambda_{j_k})} V\varphi_{j_k}^2 dV_g \to 0$.

(7) Suppose that the eigenfunctions of $\Delta_g$ are quantum ergodic in the sense that diagonal matrix elements tend to their means. What can be concluded about the dynamics of the geodesic flow?

HAS 3.2. **Hassell's scarring result for stadia.** This section is an exposition of Hassell's scarring result for the Bunimovich stadium. We follow [Has] and [Ze04].

A stadium is a domain $X = R \cup W \subset \mathbb{R}^2$ which is formed by a rectangle $R = [-\alpha, \alpha]_x \times [-\beta, \beta]_y$ and where $W = W_{-\beta} \cup W_\beta$ are half-discs of radius $\beta$ attached at either end. We fix the height $\beta = \pi/2$ and let $\alpha = t\beta$ with $t \in [1, 2]$. The resulting stadium is denoted $X_t$.

It has long been suspected that there exist exceptional sequences of eigenfunctions of $X$ which have a singular concentration on the set of "bouncing ball" orbits of $R$. These are the vertical orbits in the central rectangle that repeatedly bounce orthogonally against the flat part of the boundary. The unit tangent vectors to the orbits define an invariant Lagrangian submanifold with boundary $\Lambda \subset S^*X$. It is easy to construct approximate eigenfunctions which concentrate microlocally on this Lagrangian submanifold. Namely, let $\chi(x)$ be a smooth cutoff supported in the central rectangle and form $v_n = \chi(x)\sin ny$. Then for any pseudo-differential operator $A$ properly supported in $X$,

$$\langle Av_n, v_n\rangle \to \int_\Lambda \sigma_A \chi d\nu$$

where $d\nu$ is the unique normalized invariant measure on $\Lambda$.

Numerical studies suggested that there also existed genuine eigenfunctions with the same limit. Recently, A. Hassell has proved this to be correct for almost all stadia.

HH **Theorem 3.2.** *The Laplacian on $X_t$ is not QUE for almost every $t \in [1, 2]$.*

We now sketch the proof and develop related ideas on quantum unique ergodicity. The main idea is that the existence of the scarring bouncing ball quasi-modes implies that either

- There exist actual modes with a similar scarring property, or
- The spectrum has exceptional clustering around the bouncing ball quasi-eigenvalues $n^2$.

Hassell then proves that the second alternative cannot occur for most stadia. We now explain the ideas in more detail.

We first recall that a quasi-mode $\{\psi_k\}$ is a sequence of $L^2$-normalized functions which solve

$$\|(\Delta - \mu_k^2)\psi_k\|_{L^2} = O(1),$$

for a sequence of quasi-eigenvalues $\mu_k^2$. By the spectral theorem it follows that there must exist true eigenvalues in the interval $[\mu_k^2 - K, \mu_k^2 + K]$ for some $K > 0$. Moreover, if $\tilde{E}_{k,K}$ denotes the spectral projection for $\Delta$ corresponding to this interval, then

$$\|\tilde{E}_{k,K}\psi_k - \psi_k\|_{L^2} = O(K^{-1}).$$

To maintain consistency with our use of frequencies $\mu_k$ rather than energies $\mu_k^2$, we re-phrase this in terms of the projection $E_{k,K}$ for $\sqrt{-\Delta}$ in the interval $[\sqrt{\mu_k^2 - K}, \sqrt{\mu_k^2 + K}]$. For fixed $K$, this latter interval has width $\frac{K}{\mu_k}$.

Given a quasimode $\{\psi_k\}$, the question arises of how many true eigenfunctions it takes to build the quasi-mode up to a small error.

ESS **Definition 3.3.** We say that a quasimode $\{\psi_k\}$ of order 0 with $\|\psi_k\|_{L^2} = 1$ has $n(k)$ essential frequencies if

PSIK (16)
$$\psi_k = \sum_{j=1}^{n(k)} c_{kj}\varphi_j + \eta_k, \qquad \|\eta_k\|_{L^2} = o(1).$$

To be a quasi-mode of order zero, the frequencies $\lambda_j$ of the $\varphi_j$ must come from an interval $[\mu_k - \frac{K}{\mu_k}, \mu_k + \frac{K}{\mu_k}]$. Hence the number of essential frequencies is bounded above by the number $n(k) \leq N(k, \frac{K}{k})$ of eigenvalues in the interval. Weyl's law for $\sqrt{-\Delta}$ allows considerable clustering and only gives $N(k, \frac{K}{k}) = o(k)$ in the case where periodic orbits have measure zero. For instance, the quasi-eigenvalue might be a true eigenvalue with multiplicity saturating the Weyl bound. But a typical interval has a uniformly bounded number of $\Delta$-eigenvalues in dimension 2 or equivalently a frequency interval of with $O(\frac{1}{\mu_k})$ has a uniformly bounded number of frequencies. The dichotomy above reflects the dichotomy as to whether exceptional clustering of eigenvalues occurs around the quasi-eigenvalues $n^2$ of $\Delta$ or whether there is a uniform bound on $N(k, \delta)$.

BOUNDED **Proposition 3.4.** *If there exists a quasi-mode $\{\psi_k\}$ of order 0 for $\Delta$ with the properties:*

- *(i) $n(k) \leq C$, $\forall\, k$;*
- *(ii) $\langle A\psi_k, \psi_k\rangle \to \int_{S^*M} \sigma_A d\mu$ where $d\mu \neq d\mu_L$.*

*Then $\Delta$ is not QUE.*

The proof is based on the following lemma pertaining to near off-diagonal Wigner distributions. It gives an "everywhere" version of the off-diagonal part of Theorem 0.2 (2).

**Lemma 3.5.** *Suppose that $G^t$ is ergodic and $\Delta$ is QUE. Suppose that $\{(\lambda_{i_r}, \lambda_{j_r}),\ i_r \neq j_r\}$ is a sequence of pairs of eigenvalues of $\sqrt{-\Delta}$ such that $\lambda_{i_r} - \lambda_{j_r} \to 0$ as $r \to \infty$. Then $dW_{i_r, j_r} \to 0$.*

*Proof.* Let $\{\lambda_i, \lambda_j\}$ be any sequence of pairs with the gap $\lambda_i - \lambda_j \to 0$. Then by Egorov's theorem, any weak* limit $d\nu$ of the sequence $\{dW_{i,j}\}$ is a measure invariant under the geodesic flow. The weak limit is defined by the property that

$$\langle A^* A \varphi_i, \varphi_j \rangle \to \int_{S^*M} |\sigma_A|^2 d\nu. \tag{17}$$

If the eigenfunctions are real, then $d\nu$ is a real (signed) measure.

We now observe that any such weak* limit must be a constant multiple of Liouville measure $d\mu_L$. Indeed, we first have:

$$|\langle A^* A \varphi_i, \varphi_j \rangle| \leq |\langle A^* A \varphi_i, \varphi_i \rangle|^{1/2}\, |\langle A^* A \varphi_j, \varphi_j \rangle|^{1/2}. \tag{18}$$

Taking the limit along the sequence of pairs, we obtain

$$|\int_{S^*M} |\sigma_A|^2 d\nu| \leq \int_{S^*M} |\sigma_A|^2 d\mu_L. \tag{19}$$

It follows that $d\nu << d\mu_L$ (absolutely continuous). But $d\mu_L$ is an ergodic measure, so if $d\nu = f d\mu_L$ is an invariant measure with $f \in L^1(d\mu_L)$, then $f$ is constant. Thus,

$$d\nu = C d\mu_L, \quad \text{for some constant } C. \tag{20}$$

We now observe that $C = 0$ if $\varphi_i \perp \varphi_j$ (i.e. if $i \neq j$). This follows if we substitute $A = I$ in (17), use orthogonality and (20).

□

We now complete the proof of the Proposition by arguing by contradiction. The frequencies must come from a shrinking frequency interval, so the hypothesis of the Proposition is satisfied. If $\Delta$ were QUE, we would have (in the notation of (16):

$$\begin{aligned}
\langle A\psi_k, \psi_k \rangle &= \sum_{i,j=1}^{n(k)} c_{kj} c_{ki} \langle A\varphi_i, \varphi_j \rangle + o(1) \\
&= \sum_{j=1}^{n(k)} c_{kj}^2 \langle A\varphi_j, \varphi_j \rangle + \sum_{i \neq j=1}^{n(k)} c_{kj} c_{ki} \langle A\varphi_i, \varphi_j \rangle + o(1) \\
&= \int_{S^*M} \sigma_A d\mu_L + o(1),
\end{aligned}$$

by Proposition 3.5. This contradicts (ii). In the last line we used $\sum_{j=1}^{n(k)} |c_{kj}|^2 = 1 + o(1)$, since $\|\psi_k\|_{L^2} = 1$. □

3.3. **Proof of Hassell's scarring result.** We apply and develop this reasoning in the case of the stadium. The quasi-eigenvalues of the Bunimovich stadium corresponding to bouncing ball quasi-modes are $n^2$ independently of the diameter $t$ of the inner rectangle.

By the above, it suffices to show that that there exists a sequence $n_j \to \infty$ and a constant $M$ (independent of $j$) so that there exist $\leq M$ eigenvalues of $\Delta$ in $[n_j^2 - K, n_j^2 + K]$. An somewhat different argument is given in [Has] in this case: For each $n_j$ there exists a normalized eigenfunction $u_{k_j}$ so that $\langle u_{k_j}, v_{k_j} \rangle \geq \sqrt{\frac{3}{4}M}$. It suffices to choose the eigenfunction with eigenvalue in the interval with the largest component in the direction of $v_{k_j}$. There exists one since

$$||\tilde{E}_{[n^2-K, n^2+K} v_n|| \geq \frac{3}{4}.$$

The sequence $\{u_{n_k}\}$ cannot be Liouville distributed. Indeed, for any $\varepsilon > 0$, let $A$ be a self-adjoint semi-classical pseudo-differential operator properly supported in the rectangle so that $\sigma_A \leq 1$ and so that $\|(Id - A)v_n\| \leq \varepsilon$ for large enough $n$. Then

$$\begin{aligned} \langle A^2 u_{k_j}, u_{k_j} \rangle = \|A u_{k_j}\|^2 \geq \left| \langle A u_{k_j}, v_{k_j} \rangle \right|^2 \\ = \left| \langle u_{k_j}, A v_{k_j} \rangle \right|^2 \geq \left( |\langle u_{k_j}, v_{k_j} \rangle| - \varepsilon \right)^2 \geq \left( \sqrt{\frac{3}{4}M} - \varepsilon \right)^2. \end{aligned}$$

Choose a sequence of operators $A$ such that $||(Id - A)v_n|| \to 0$ and so that the support of $\sigma_A$ shrinks to the set of bouncing ball covectors. Then the mass of any quantum limit of $\{u_{n_k}\}$ must have mass $\geq \frac{3}{4}M$ on $\Lambda$.

Thus, the main point is to eliminate the possibility of exceptional clustering of eigenvalues around the quasi-eigenvalues. In fact, no reason is known why no exceptional clustering should occur. Hassell's idea is that it can however only occur for a measure zero set of diameters of the inner rectangle. The proof is based on Hadamard's variational formula for the variation of Dirichlet or Neumann eigenvalues under a variation of a domain. In the case at hand, the stadium is varied by horizontally (but not vertically) expanding the inner rectangle. In the simplest case of Dirichlet boundary conditions, the eigenvalues are forced to decrease as the rectangle is expanded. The QUE hypothesis forces them to decrease at a uniform rate. But then they can only rarely cluster at the fixed quasi-eigenvalues $n^2$. If this ever happened, the cluster would move left of $n^2$ and there would not be time for a new cluster to arrive.

Here is a more detailed sketch. Under the variation of $X_t$ with infinitesimal variation vector field $\rho_t$, Hadamard's variational formula gives,

$$\frac{dE_j(t)}{dt} = \int_{\partial X_t} \rho_t(s) (\partial_n u_j(t)(s))^2 ds.$$

Then

$$E_j^{-1} \frac{d}{dt} E_j(t) = - \int_{\partial X_t} \rho_t(s) u_j^b(s)^2 ds.$$

Let $A(t)$ be the area of $S_t$. By Weyl's law, $E_j(t) \sim c \frac{j}{A(t)}$. Since the area of $X_t$ grows linearly, we have on average $\dot{E}_j \sim -C \frac{E_j}{A(t)}$. Quantum ergodicity gives the asymptotics individually for almost all eigenvalues. Let

$$f_j(t) = \int_{\partial X_t} \rho_t(s)|u_j^b(t;s)|^2 ds.$$

Then $\dot{E}_j = -E_j f_j$. Then Quantum ergodicity implies that $|u_j^b|^2 \to \frac{1}{A(t)}$ weakly on the boundary along a subsequence of density one. QUE is the hypothesis that this occurs for the entire sequence, i.e.

$$f_j(t) \to \frac{k}{A(t)} > 0, \quad k := \int_{\partial S_t} \rho_t(s) ds.$$

Hence,

$$\frac{\dot{E}_j}{E} = -kA(t)(1 + o(1)), \quad j \to \infty.$$

Hence there is a lower bound to the velocity with which eigenvalues decrease as $A(t)$ increases. Eigenvalues can therefore not concentrate in the fixed quasi-mode intervals $[n^2 - K, n^2 + K]$ for all $t$. But then there are only a bounded number of eigenvalues in this interval; so Proposition BOUNDED 3.4 implies QUE for the other $X_t$. A more detailed analysis shows that QUE holds for almost all $t$.

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